

A SUFFICIENT CONDITION FOR p -VALENTLY HARMONIC FUNCTIONS

TOSHIO HAYAMI

ABSTRACT. For normalized harmonic functions $f(z) = h(z) + \overline{g(z)}$ in the open unit disk \mathbb{U} , a sufficient condition on $h(z)$ for $f(z)$ to be p -valent in \mathbb{U} is discussed. Moreover, some interesting examples and images of $f(z)$ satisfying the obtained condition are enumerated.

1. INTRODUCTION AND DEFINITIONS

For a fixed p ($p = 1, 2, 3, \dots$), a meromorphic function $f(z)$ in a domain \mathbb{D} is said to be p -valent (or multivalent of order p) in \mathbb{D} if for each w_0 (infinity included) the equation $f(z) = w_0$ has at most p roots in \mathbb{D} where the roots are counted in accordance with their multiplicity and if there is some w_1 such that the equation $f(z) = w_1$ has exactly p roots in \mathbb{D} . In particular, $f(z)$ is said to be univalent (one-to-one) in \mathbb{D} when $p = 1$.

A complex-valued harmonic function $f(z)$ in \mathbb{D} is given by $f(z) = h(z) + \overline{g(z)}$ where $h(z)$ and $g(z)$ are analytic in \mathbb{D} . We call $h(z)$ and $g(z)$ the analytic part and co-analytic part of $f(z)$, respectively. A necessary and sufficient condition for $f(z)$ to be locally univalent and sense-preserving in \mathbb{D} is $|h'(z)| > |g'(z)|$ for all $z \in \mathbb{D}$ (see [2] or [6]). The theory and applications of harmonic functions are stated in a book due to Duren [3]. Let $\mathcal{H}(p)$ denote the class of functions $f(z)$ of the form

$$f(z) = h(z) + \overline{g(z)} = z^p + \sum_{n=p+1}^{\infty} a_n z^n + \overline{\sum_{n=p}^{\infty} b_n z^n} \quad (1.1)$$

which are harmonic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. We next denote by $\mathcal{S}_{\mathcal{H}}(p)$ the class of functions $f(z) \in \mathcal{H}(p)$ which are p -valent and sense-preserving in \mathbb{U} . Then, we say that $f(z) \in \mathcal{S}_{\mathcal{H}}(p)$ is a p -valently harmonic function in \mathbb{U} .

In the present paper, we discuss a sufficient condition about $h(z)$ for $f(z) \in \mathcal{H}(p)$ given by (1.1), satisfying

$$g'(z) = z^{m-1} h'(z) \quad (1.2)$$

for some m ($m = 2, 3, 4, \dots$), to be in the class $\mathcal{S}_{\mathcal{H}}(p)$. In this case, we can write

$$f(z) = h(z) + \overline{z^{m-1} h(z) - (m-1) \int_0^z \zeta^{m-2} h(\zeta) d\zeta} \quad (1.3)$$

which means that $f(z)$ is well defined if $h(z)$ is given.

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2. MAIN RESULT

Our result is contained in

Theorem 2.1. *Let $h(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$ be analytic in the closed unit disk $\overline{\mathbb{U}} = \{z \in \mathbb{C} : |z| \leq 1\}$ with $H(z) = h'(z)/z^{p-1} \neq 0$ ($z \in \overline{\mathbb{U}}$) and let*

$$F(t) = (2p + m - 1)t + 2 \arg(H(e^{it})) \quad (-\pi \leq t < \pi) \quad (2.1)$$

for some m ($m = 2, 3, 4, \dots$). If for each $k \in K = \{0, \pm 1, \pm 2, \dots, \pm [\frac{2p+m+1}{2}]\}$ where $[\]$ is the Gauss symbol, the equation

$$F(t) = 2k\pi \quad (2.2)$$

has at most a single root in $[-\pi, \pi)$ and for all $k \in K$ there exist exactly $2p + m - 1$ such roots in $[-\pi, \pi)$, then the harmonic function $f(z) = h(z) + \overline{g(z)}$ with $g'(z) = z^{m-1}h'(z)$ belongs to the class $\mathcal{S}_{\mathcal{H}}(p)$ and maps \mathbb{U} onto a domain surrounded by $2p + m - 1$ concave curves with $2p + m - 1$ cusps.

Proof. We first consider the function $\varphi(t)$ defined as

$$\varphi(t) = f(e^{it}) = h(e^{it}) + \overline{g(e^{it})} \quad (-\pi \leq t < \pi). \quad (2.3)$$

Supposing that

$$\varphi'(t) = i \left(zh'(z) - \overline{zg'(z)} \right) = iz \left(h'(z) - \overline{z^{m+1}h'(z)} \right) = 0 \quad (z = e^{it}), \quad (2.4)$$

we need the following equation

$$z^{m+1} \frac{h'(z)}{h'(z)} = 1 \quad \left(\overline{z} = e^{-it} = \frac{1}{z} \right). \quad (2.5)$$

This means that

$$\begin{aligned} \arg \left(e^{i(m+1)t} \frac{h'(e^{it})}{\overline{h'(e^{it})}} \right) &= (m+1)t + 2 \arg \left(e^{i(p-1)t} H(e^{it}) \right) \\ &= (2p + m - 1)t + 2 \arg(H(e^{it})) = 2k\pi \end{aligned} \quad (2.6)$$

for some $k \in K$ which gives us the equation (2.2). In consideration of the assumption of the theorem, there are $2p + m - 1$ distinct roots on the unit circle $\partial\mathbb{U} = \{z \in \mathbb{C} : |z| = 1\}$ and they divide $\partial\mathbb{U}$ onto $2p + m - 1$ arcs. Moreover, since $g''(z) = (m-1)z^{m-2}h'(z) + z^{m-1}h''(z)$, we obtain that

$$\begin{aligned} \varphi''(t) &= - \left(zh'(z) + z^2 h''(z) + \overline{zg'(z)} + \overline{z^2 g''(z)} \right) \\ &= - \left(zh'(z) + z^2 h''(z) + m \overline{z^m h'(z)} + \overline{z^{m+1} h''(z)} \right) \end{aligned} \quad (2.7)$$

and

$$\begin{aligned}
\varphi''(t)\overline{\varphi'(t)} &= -\left(zh'(z) + z^2h''(z) + m\bar{z}^m\overline{h'(z)} + \bar{z}^{m+1}\overline{h''(z)}\right)(-i)\left(\bar{z}h'(z) - z^mh'(z)\right) \\
&= i\left(-(m-1)|h'(z)|^2 + z\overline{h'(z)}h''(z) - \overline{zh'(z)h''(z)} - z^{m+1}h'(z)^2\right. \\
&\quad \left.+ \overline{z^{m+1}h'(z)^2} + (m-1)\bar{z}^{m+1}\overline{h'(z)}^2 - z^{m+2}h'(z)h''(z) + \overline{z^{m+2}h'(z)h''(z)}\right) \\
&= i\left\{(m-1)\left(\bar{z}^{m+1}\overline{h'(z)}^2 - |h'(z)|^2\right)\right. \\
&\quad \left.+ 2i\operatorname{Im}\left(z\overline{h'(z)}h''(z) - z^{m+1}h'(z)^2 - z^{m+2}h'(z)h''(z)\right)\right\}. \tag{2.8}
\end{aligned}$$

This leads us that

$$\operatorname{Im}\left(\varphi''(t)\overline{\varphi'(t)}\right) = (m-1)|h'(z)|^2\operatorname{Re}\left\{\bar{z}^{m+1}\left(\frac{\overline{h'(z)}}{|h'(z)|}\right)^2 - 1\right\} \leq 0 \tag{2.9}$$

for all $z = e^{it}$. Therefore, it follows that

$$\operatorname{Im}\left(\frac{\varphi''(t)}{\varphi'(t)}\right) = \frac{1}{|\varphi'(t)|^2}\operatorname{Im}\left(\varphi''(t)\overline{\varphi'(t)}\right) \leq 0 \tag{2.10}$$

which shows that $\varphi(t)$ maps $[-\pi, \pi)$ onto a union of $2p + m - 1$ concave curves. By the help of a simple geometrical observation, we know that the image of $\partial\mathbb{U}$ as a union of $2p + m - 1$ concave arcs is a simple curve. Thus, $f(z)$ is p -valent in \mathbb{U} and the image $f(\mathbb{U})$ is a domain surrounded by $2p + m - 1$ concave curves with $2p + m - 1$ cusps. \square

Remark 2.1. If we take $p = 1$ in Theorem 2.1, then we readily arrive at the univalence criterion for harmonic functions due to Hayami and Owa [5, Theorem 2.1] (see also [7]).

3. SOME ILLUSTRATIVE EXAMPLES AND IMAGE DOMAINS

In this section, we discuss functions $f(z) = h(z) + \overline{g(z)}$ satisfying the conditions of Theorem 2.1 and their image domains.

Example 3.1. Let $h(z) = z^p$. Then we easily see that the equation (2.2) becomes

$$(2p + m - 1)t = 2k\pi \quad \left(k = 0, \pm 1, \pm 2, \dots, \pm \left[\frac{2p + m + 1}{2}\right]\right) \tag{3.1}$$

which satisfies the conditions of Theorem 2.1. Hence, the function

$$f(z) = h(z) + \overline{g(z)} = z^p + \frac{p}{p + m - 1} \overline{z^{p+m-1}} \quad (g'(z) = z^{m-1}h'(z)) \tag{3.2}$$

belongs to the class $\mathcal{S}_{\mathcal{H}}(p)$ and it maps \mathbb{U} onto a domain surrounded by $2p + m - 1$ concave curves with $2p + m - 1$ cusps. Taking $p = 2$ and $m = 4$ for example, we know that the function

$$f(z) = z^2 + \frac{2}{5}\bar{z}^5 \tag{3.3}$$

is a 2-valently harmonic function in \mathbb{U} and it maps \mathbb{U} onto the domain surrounded by 7 concave curves with 7 cusps as shown in Figure 1.

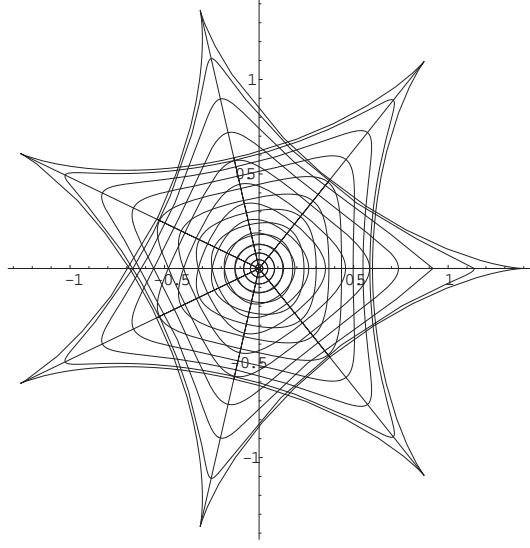


FIGURE 1. The image of $f(z) = z^2 + \frac{2}{5}z^5$.

Remark 3.1. Since we can rewrite $F(t)$ given by (2.1) as follows:

$$F(t) = (2p + m - 1)t + 2\operatorname{Im}(\log H(e^{it})), \quad (3.4)$$

we have that

$$\begin{aligned} F'(t) &= 2p + m - 1 + 2\operatorname{Re}\left(\frac{e^{it}h''(e^{it})}{h'(e^{it})} - (p - 1)\right) \\ &= m + 1 + 2\operatorname{Re}\left(\frac{e^{it}h''(e^{it})}{h'(e^{it})}\right) \end{aligned} \quad (3.5)$$

which implies that $F(t)$ is increasing if

$$\operatorname{Re}\left(1 + \frac{zh''(z)}{h'(z)}\right) > -\frac{m-1}{2} \quad (z \in \mathbb{U}). \quad (3.6)$$

By the above remark, we derive the following example.

Example 3.2. Let $h(z) = z^p + \frac{c}{p+1}z^{p+1}$ $\left(|c| \leq p - \frac{2p}{2p+m+1}\right)$. Then the equation (2.2) becomes

$$F(t) = (2p + m - 1)t + 2\arg(p + ce^{it}). \quad (3.7)$$

Noting

$$\begin{aligned} \operatorname{Re}\left(1 + \frac{zh''(z)}{h'(z)}\right) &= p + 1 - \operatorname{Re}\left(\frac{p}{p + cz}\right) \\ &> p + 1 - \frac{p}{p - |c|} \geq -\frac{m-1}{2} \quad (z \in \mathbb{U}), \end{aligned} \quad (3.8)$$

$$F(-\pi) = -(2p + m - 1)\pi - 2 \arctan \left(\frac{|c| \sin \theta}{p - |c| \cos \theta} \right) \quad (3.9)$$

and

$$F(\pi) = (2p + m - 1)\pi - 2 \arctan \left(\frac{|c| \sin \theta}{p - |c| \cos \theta} \right) \quad (3.10)$$

where $0 \leq \theta = \arg(c) < 2\pi$, we see that $F(t)$ satisfies the conditions of Theorem 2.1. Hence, the function

$$f(z) = h(z) + \overline{g(z)} = z^p + \frac{c}{p+1}z^{p+1} + \overline{\frac{p}{p+m-1}z^{p+m-1} + \frac{c}{p+m}z^{p+m}} \quad (3.11)$$

belongs to the class $\mathcal{S}_{\mathcal{H}}(p)$ and it maps \mathbb{U} onto a domain surrounded by $2p + m - 1$ concave curves with $2p + m - 1$ cusps. Putting $p = 3$, $m = 2$ and $c = i$ ($|c| = 1 < \frac{7}{3}$), we know that the function

$$f(z) = z^3 + \frac{i}{4}z^4 + \frac{3}{4}\overline{z}^4 - \frac{i}{5}\overline{z}^5 \quad (3.12)$$

is a 3-valently harmonic function in \mathbb{U} and it maps \mathbb{U} onto the domain surrounded by 7 concave curves with 7 cusps as shown in Figure 2. We check only the boundary of $f(\mathbb{U})$, for the sake of simplicity.

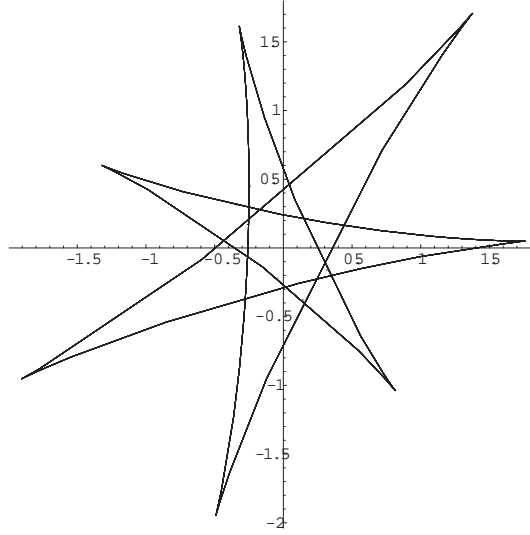


FIGURE 2. The image of $f(z) = z^3 + \frac{i}{4}z^4 + \frac{3}{4}\overline{z}^4 - \frac{i}{5}\overline{z}^5$.

4. APPENDIX

The next result was conjectured by Mocanu [8] and proved by Bshouty and Lyzzaik [1].

Theorem 4.1. *If $h(z)$ and $g(z)$ are analytic in \mathbb{U} , with $h'(0) \neq 0$, which satisfy*

$$g'(z) = zh'(z) \quad (4.1)$$

and

$$\operatorname{Re} \left(1 + \frac{zh''(z)}{h'(z)} \right) > -\frac{1}{2} \quad (4.2)$$

for all $z \in \mathbb{U}$, then the harmonic function $f(z) = h(z) + \overline{g(z)}$ is univalent close-to-convex in \mathbb{U} .

We immediately notice that the above theorem is closely related to Theorem 2.1 and Remark 3.1 with $p = 1$ and $m = 2$. This motivates us to state

Conjecture 4.1. *If the function $f(z)$ given by (1.1) is harmonic in \mathbb{U} which satisfies*

$$g'(z) = z^{m-1}h'(z) \quad (4.3)$$

and

$$\operatorname{Re} \left(1 + \frac{zh''(z)}{h'(z)} \right) > -\frac{m-1}{2} \quad (z \in \mathbb{U}) \quad (4.4)$$

for some m ($m = 2, 3, 4, \dots$), then $f(z)$ is p -valent in \mathbb{U} .

Finally, in view of the process of proving Theorem 2.1, we obtain the following interesting example.

Example 4.1. *If we consider special functions $h(z)$ and $g(z)$ given by*

$$h'(z) = \frac{pz^{p-1}}{1+z^{2p+m-1}} \quad \text{and} \quad g'(z) = \frac{pz^{p+m-2}}{1+z^{2p+m-1}} \quad (g'(z) = z^{m-1}h'(z)), \quad (4.5)$$

then the function $\varphi(t)$ given by (2.3) satisfies

$$\begin{aligned} \varphi'(t) &= iz \left(h'(z) - \overline{z^{m+1}h'(z)} \right) \\ &= iz \left(\frac{pz^{p-1}}{1+z^{2p+m-1}} - \frac{p\bar{z}^{p+m}}{1+\bar{z}^{2p+m-1}} \right) \\ &= iz \left(\frac{pz^{p-1}}{1+z^{2p+m-1}} - \frac{pz^{p-1}}{1+z^{2p+m-1}} \right) = 0 \quad (z = e^{it}) \end{aligned} \quad (4.6)$$

for any $z^{2p+m-1} = e^{i(2p+m-1)t} \neq -1$. Thus, the function

$$f(z) = h(z) + \overline{g(z)} = \int_0^z \frac{p\zeta^{p-1}}{1+\zeta^{2p+m-1}} d\zeta + \overline{\int_0^z \frac{p\zeta^{p+m-2}}{1+\zeta^{2p+m-1}} d\zeta} \quad (4.7)$$

is a member of the class $\mathcal{S}_{\mathcal{H}}(p)$ and it maps \mathbb{U} onto a domain surrounded by $2p+m-1$ straight lines with $2p+m-1$ cusps. Indeed, setting $p=2$ and $m=2$, we know that

$$f(z) = \int_0^z \frac{2\zeta}{1+\zeta^5} d\zeta + \overline{\int_0^z \frac{2\zeta^2}{1+\zeta^5} d\zeta} \quad (4.8)$$

is a 2-valently harmonic function and it maps \mathbb{U} onto a star as shown in Figure 3. Furthermore, if we take $p=1$ in (4.7), then we see that the function

$$f_{m+1}(z) = h(z) + \overline{g(z)} = \int_0^z \frac{1}{1+\zeta^{m+1}} d\zeta + \overline{\int_0^z \frac{\zeta^{m-1}}{1+\zeta^{m+1}} d\zeta} \quad (4.9)$$

is univalent in \mathbb{U} and it maps \mathbb{U} onto a $(m+1)$ -sided polygon. For example, the function

$$f_8(z) = \int_0^z \frac{1}{1+\zeta^8} d\zeta + \overline{\int_0^z \frac{\zeta^6}{1+\zeta^8} d\zeta} \quad (m=7) \quad (4.10)$$

maps \mathbb{U} onto an octagon (schlicht domain) as shown in Figure 4.

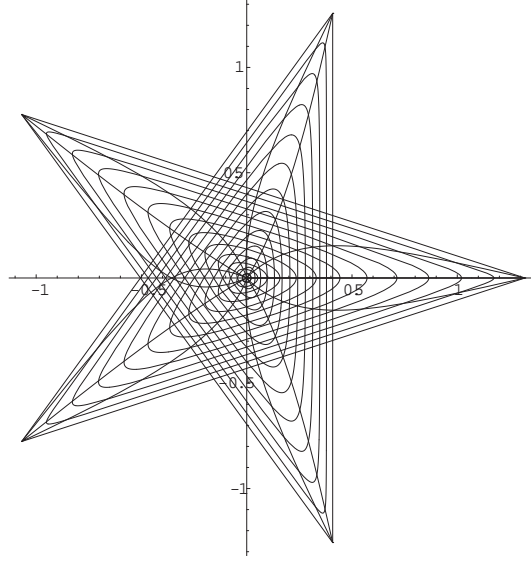


FIGURE 3. The image of $f(z) = \int_0^z \frac{2\zeta}{1+\zeta^5} d\zeta + \overline{\int_0^z \frac{2\zeta^2}{1+\zeta^5} d\zeta}$.

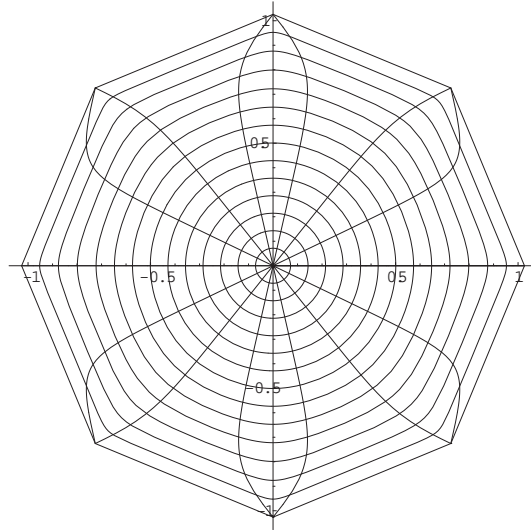


FIGURE 4. The image of $f_8(z) = \int_0^z \frac{1}{1+\zeta^8} d\zeta + \overline{\int_0^z \frac{\zeta^6}{1+\zeta^8} d\zeta}$.

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